Mesoscopic description of the annealed Ising model, and multiplicative noise

Walter Genovese,[1] Miguel A. Muñoz,[1,2] and P. L. Garrido[2]
[1]*Dipartimento di Fisica, Università di Roma ''La Sapienza,'' Piazzale Aldo Moro 2, I-00185 Roma, Italy*
[2]*Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain*
(Received 5 June 1998)

A new type of Langevin equation exhibiting a nontrivial phase transition associated with the presence of multiplicative noise is discussed. The equation is derived as a mesoscopic representation of the microscopic annealed Ising model (AIM) proposed by Thorpe and Beeman, and reproduces perfectly its basic phenomenology. The AIM exhibits a nontrivial behavior as the temperature is increased, in particular it presents a disorder-to-order phase transition at low temperatures, and an order-to-disorder transition at higher temperatures. This behavior resembles that of some Langevin equations with multiplicative noise, which exhibit also two analogous phase transitions as the noise amplitude is increased. By comparing the standard models for noise-induced transitions with our new Langevin equation we elucidate that the mechanisms controlling the disorder-to-order transitions in both of them are essentially different, even though for both of them the presence of multiplicative noise is a key ingredient. [S1063-651X(98)06911-6]

PACS number(s): 05.40.+j

I. INTRODUCTION

A great deal of attention has been recently devoted to the study of physical effects induced by the presence of noise, i.e., phenomena appearing in stochastic systems, which would be absent in the sole presence of the deterministic part of the corresponding Langevin equation [1]. By now it is clear that noise can generate quite unexpected and counterintuitive behaviors as, for example, *stochastic resonance* [2], in which the output to input ratio of a bistable system subjected to the presence of an oscillating force is strongly enhanced by the presence of an additional stochastic term. Other examples are the resonant activation [3], and the noise induced spatial patterns [4]. Another type of phenomenon the noise is at the base of are the so called *noise induced phase transitions*. These came to light in an interesting paper by Van den Broek, Parrondo, and Toral [5] (see also [1,6,7]). These authors pointed out the fact that some Langevin equations may exhibit a noise-induced ordering transition (NIOT), i.e., a phase transition that is not expected from the analysis of the deterministic part of such equation. The phenomenology is as follows.

(i) For low enough noise amplitudes the system is disordered (i.e., the order parameter takes a zero value).

(ii) At a certain critical value of the noise amplitude the system exhibits a NIOT and, in a range of noise intensities above it, the system remains ordered.

(iii) Finally, for noise amplitudes larger than a second critical value, the noise operates in a more standard way, disordering the system again. We refer to this second phase transition as noise induced disordering transition (NIDT).

A physical explanation of the NIOT was proposed in [5]; the ordering of the system is the consequence of the interplay between the noise and the spatial coupling [8]. In particular, the noise generates a short time instability at every single site, and the presence of a spatial coupling renders stable the nontrivial state generated in that way.

A minimal model capturing the essence of the NIOT has been recently proposed [9]. It has been clarified that the essence of the NIOT is purely multiplicative, this is, in order to generate an ordering transition the noise has to appear multiplied by the field variable. In this way, it has been possible to recognize that the NIOT is characterized by a set of critical exponents other than those of well established universality classes (as, for example, that of the Ising model) [10,9]. It has also been shown that due to the multiplicative origin of this transition it is possible to observe the phenomena in $d=1$, a dimension at which it is very unusual to observe phase transitions.

Other results concerning NIOTs and NIDTs can be found in the literature [11–13]. A common feature of all the previously referred models is that they are defined by means of Langevin equations, that is, equations describing the physics at a mesoscopic, coarse-grained scale (in fact, the concept of noise is meaningful only at this level). In this context, analyzing microscopic models that exhibit similar nontrivial behaviors is an interesting task; one of these models is the annealed Ising model [14]. By studying the connection between microscopic systems and their respective mesoscopic representation one could shed some light on the way in which microscopic mechanisms generate the very nontrivial effects described at a mesoscopic scale.

In what follows we introduce the time honored annealed Ising model. It was proposed and described more than twenty years ago by Thorpe and Beeman [14]. A more detailed description of it will be presented in the next section; here we summarize the main properties we are interested in. The system is an Ising model in which the interactions J among spins are annealed (not quenched) random variables that change from bond to bond and are extracted from a fixed probability distribution, $P(J)$. Under certain conditions (this is, for some distributions $P(J)$ to be specified later), the system phenomenology is as follows: (i) For low temperatures the system is disordered, i.e., the averaged magnetization is zero. (ii) At a critical value of the temperature, T_1, the system exhibits a second order phase transition. As the temperature is further increased above T_1 the averaged magnetization keeps on growing until it reaches a maximum value

and it starts decreasing if T is increased further. (iii) At a second critical temperature, T_2, the system exhibits another phase transition (analogous to the well known ferromagnetic-paramagnetic disordering transition of the standard pure Ising model). The system remains disordered for temperatures higher than T_2.

This phenomenology resembles very much the behavior of the previously described noise induced transitions in Langevin equations. It is our purpose here to analytically derive a coarse-grained, mesoscopic representation, in terms of a Langevin equation, of the microscopic annealed Ising model (AIM) to further explore the eventual relations between both phenomena.

II. ANNEALED ISING MODEL

Let us consider a d-dimensional impure Ising model in the sense that the value of the coupling constant among spins J changes from bond to bond, being an annealed random variable with a fixed temperature-independent probability distribution, $P(J)$ (which is not quenched but annealed at every site). Following the strategy proposed by Thorpe and Beeman [14] the model can be exactly mapped into a standard pure Ising model with an effective parameter, $K=J/T$, that depends on $P(J)$ and T, and we write as $K_{\mathrm{eff}}(T)$. In particular [14],

$$\int dJ \frac{P(J)}{\coth[K_{\mathrm{eff}}-J/T]-\epsilon(K_{\mathrm{eff}})}=0, \tag{1}$$

where $\epsilon(K)$ is the correlation function of two nearest-neighbor spins in the pure Ising model. By solving the implicit equation (1) one obtains K_{eff} as a function of the temperature and the parameters characterizing $P(J)$. Note that, in particular, for the two-dimensional case, the Onsager's solution [15] provides an explicit value for $\epsilon(K)$ and therefore Eq. (1) can be solved and, furthermore, the system magnetization can be expressed as a function of T. Let us suppose now that, in particular, the distribution $P(J)$ is centered at a positive value J_0 (favoring ferromagnetic ordering), and has a variable width (standard deviation), δJ. The resulting magnetization for this particular type of distribution is qualitatively represented in Fig. 1 (see also [14]).

Observe that for narrow distributions of J the magnetization curve is similar to its counterpart in the pure Ising model. Instead, as δJ is increased, a disordering tendency is observed at low temperatures, and in particular, for large values of the width (as for example, δJ_4 in Fig. 1) the system is disordered at low temperatures, and exhibits a disorder-to-order phase transition at a certain temperature. The standard ferromagnetic-paramagnetic (order-to-disorder) transition is also present and occurs at a variable value of T for different values of δJ.

The physical mechanism leading to the previous behavior was argued in [14] to be the competition between ferromagnetic and antiferromagnetic types of interactions that emerges when sufficiently large values of δJ are considered. In particular, when $\delta J > J_0$ both positive and negative values of the coupling constant are accessible at each bond, and in that case, for low temperatures, the system is in a *frustrated state* in which ferromagnetic and antiferromagnetic domains

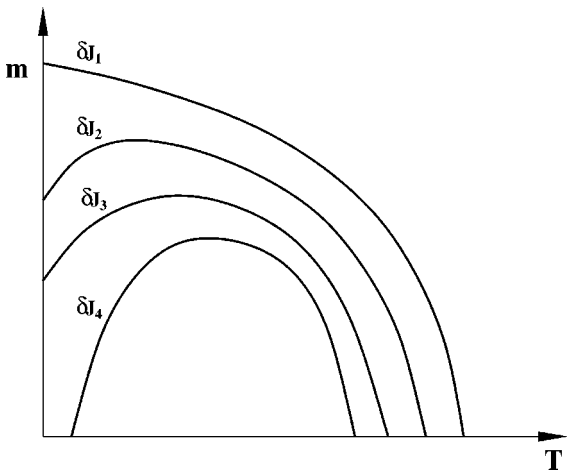

FIG. 1. Magnetization as a function of the temperature T, for different values of δJ; $0=\delta J_1<\delta J_2<\delta J_3<\delta J_4$ for the annealed Ising model.

compete. That frustration makes the ferromagnetic order parameter vanish. As the temperature is further raised the thermal noise activates annihilation of domain walls and the system is more likely to ordinate. As a result, the averaged magnetization grows with increasing temperature. At a given point this effect ceases, and the standard role of the temperature as a disorganizing source sets to work.

III. CONTINUOUS REPRESENTATION

Let us now follow a standard procedure [16] to cast the previous AIM into a continuous Langevin equation. For that purpose we first consider the pure Ising model case, and write down its associated equilibrium partition function:

$$Z=\sum_{\{s\}} \exp\left(\sum_{ij} K_{ij}s_i s_j\right). \tag{2}$$

Introducing auxiliary Gaussian integrals in terms of continuous variables ϕ_i (with i varying from 1 to the total number of spins, N, in the lattice), and performing the change of variables $\psi_i=K_{ij}^{-1}\phi_j$ we obtain [16]

$$Z\propto \int d\psi_1 d\psi_2 \cdots d\psi_N \times \exp\left[-\frac{1}{4}\psi_i K_{ij}\psi_j+\sum_i \ln\cosh(K_{ij}\psi_j)\right]. \tag{3}$$

Expanding the hyperbolic cosine in power series, performing a transformation to Fourier space, considering only the leading dependence on the temperature, and performing the continuous limit we finally obtain [16]

$$Z\propto \int d[\psi]e^{-H},$$

$$H=\tfrac{1}{4}\int d^dx[K_0(1-2K_0)\psi^2(x)+\rho(4K_0-1)(\nabla\psi)^2 + \tfrac{1}{3}K_0^4\psi^4(x)] \tag{4}$$

with $K_0=\int d^dx K(x)$, and $\rho=1/2\int d^dx K(x)x^2$. In this way we have derived a Ginzsburg-Landau coarse grained Hamil-

tonian for the Ising model. This could have been guessed *a priori* by using heuristic arguments, but we have preferred to follow the previous procedure that permits one to obtain explicit expressions for the coefficients as a function of the microscopic parameters. In this way, observe, for example, that both the diffusion constant and the coefficient of the quadratic term depend on the coupling through K_0; therefore in order to simplify the notation we define the diffusion constant $D=\rho(4K_0-1)$. Taking only the main relevant dependences on D we can write

$$H=\int d^dx\left[\frac{aD}{2}\psi^2+\frac{b}{4}\psi^4+\frac{D}{2}(\nabla\psi)^2\right], \tag{5}$$

where a is a tuning parameter proportional to the distance to the critical temperature, and b is a positive parameter. Let us stress once more that we are neglecting higher-order dependences of b and a on D, and we assume them to be unessential to reproduce the microscopic phenomenology of interest at the mesoscopic level (this hypothesis will be verified afterwards). The simplest Langevin equation with a stationary distribution characterized by a Gibbsian distribution with the Hamiltonian in Eq. (5) is well known to be [17,18]

$$\partial_t\psi=-(aD+b\psi^2)\psi+D\nabla^2\psi+\eta(t) \tag{6}$$

where $\eta(t)$ is a Gaussian white noise with $\langle\eta(x,t)\rangle=0$, and $\langle\eta(x,t)\,\eta(x',t')\rangle=\delta^d(x-x')\,\delta(t-t')$.

At this point we can analyze the effects of an annealed distribution of J in the microscopic AIM at the level of Langevin equations. For that purpose let us observe that in order to mimic the variability of the coupling in the AIM we can just substitute D at each site in Eq. (6) by a stochastic variable, namely, $D\to D+\xi(x,t)$, with $\langle\xi(x,t)\rangle=0$ and $\langle\xi(x,t)\xi(x',t')\rangle=\sigma_D^2\delta^d(x-x')\,\delta(t-t')$, where D and σ_D play the role of J_0 and δJ, respectively, in the microscopic model. In this way we obtain

$$\partial_t\psi=-[a(D+\xi)+b\psi^2]\psi+D\nabla^2\psi+\nabla(\xi\nabla\psi)+\eta(t). \tag{7}$$

This equation (intended in the Ito interpretation [18]) constitutes our continuous representation of the AIM. Let us underline that there are two differences with respect to the pure case, Eq. (6), the presence of a *multiplicative noise*, and an extra term that couples spatial fluctuations of D with $\nabla\psi$. Changes of a, the parameter that appears by multiplying both the linear term and the multiplicative noise, correspond to temperature variations.

We have analyzed Eq. (7) in mean field approximation [8,19,9], and by performing systematic numerical simulations in two dimensions. The mean field approximation is performed along the lines discussed in [8,19,9]. For the numerical simulation we have employed the Euler method [19], in a 32×32 lattice, with lattice spacing $\Delta a=1$, and considered a time mesh $\Delta t=0.001$. Without loss of generality the parameters b and D have been fixed to 1 and 10, respectively. Different noise amplitudes, σ_D, have been considered. The main results we have obtained are as follows: in both the mean field approximation and in the numerical simulation, we reproduce the qualitative behavior of the order parameter as a function of the temperature characteristic of the microscopic model (see Figs. 2 and 3 and compare them with Fig. 1).

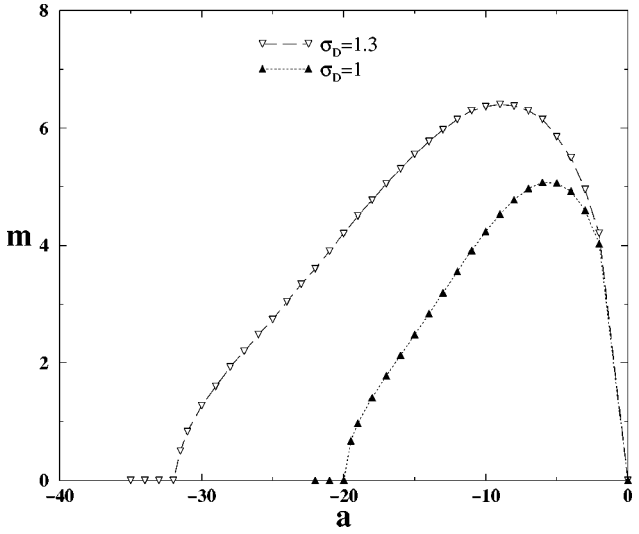

FIG. 2. Averaged magnetization of the Langevin equation as a function of a (i.e., the temperature), in mean field approximation, for different values of σ_D.

In the mean field approximation the order-to-disorder critical point is located at $a=0$, and numerically we obtain also a critical value close to zero that does not depend on σ_D. On the contrary, the location of the disorder-to-order transition depends on σ_D, analogously as the location of T_1 depends on δJ in the AIM. Observe that this transition is not sharp in the lowermost curve of Fig. 3 due to finite size effects. Curves in Fig. 2 and Fig. 3 change with increasing σ_D in the same way as they do in the AIM when increasing δJ, i.e., the larger the noise the smaller the ordering.

Let us stress once more that in order to obtain the transition, we change both the coefficient of the linear term and of the multiplicative noise term. If one of these two coefficients was kept fixed while the other was changed the microscopic

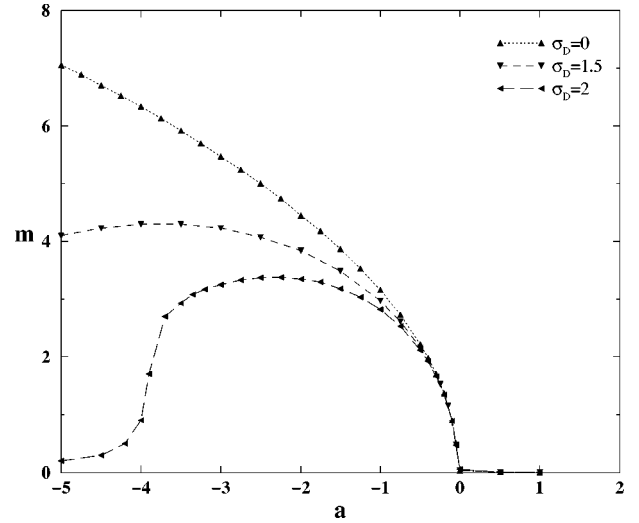

FIG. 3. Averaged magnetization of the Langevin equation as a function of a (i.e., the temperature), in numerical simulations, for different values of σ_D.

phenomenology would not be reproduced. *The presence of the multiplicative noise term is essential to generate the disorder-to-order transition*. We have performed a numerical study of Eq. (7) omitting the term proportional to $\nabla(\xi\nabla\psi)$, and conclude that none of the previous conclusions is qualitatively affected by this suppression; by omitting this term the disorder-to-order critical point is shifted to a lower value of a, and consequently this term has only a disorganizing effect. We could consequently write down a minimal model just by dropping out this unnecessary term, in the same way we omitted other irrelevant higher order dependences on D in the derivation of the Langevin equation. We conclude that *the proposed Langevin equation with multiplicative noise in the Ito representation reproduces qualitatively all the interesting properties of the anneal Ising model, and in particular the reentrant phase transition.* Therefore, once more it is shown that the multiplicative noise is the key ingredient of highly nontrivial phenomena in stochastic systems at a mesoscopic level.

Let us finally remark that the phenomenon we have just described *is not* the usual noise induced transition as reported in previous works [5,11,9]. First of all, in those works only the multiplicative noise amplitude has to be changed to obtain a NIOT, while in our case the transition is obtained by varying the parameter a that multiplies both the multiplicative noise and the linear term. Consequently in our case the disorder-to-order transition is not purely noise induced. Second, considering a Stratonovich representation of the Langevin equation with multiplicative noise is essential in those works to generate noise-induced ordering. In fact, standard Langevin equations such as those described in [5,11,9] do not exhibit NIOTs when intended in the Ito representation [20]. On the other hand, in the model presented here, the Langevin equation is intended in the Ito sense, and due to its peculiar structure, namely, the coupling between a and $\xi(t)$, that we have justified from a microscopic point of view, it can exhibit a rather rich phenomenology. In particular the system shows an ordering and a disordering transition as the temperature is increased but it does not exhibit, for example, the short time instability characteristic of the phenomena discussed in [5,8,9].

ACKNOWLEDGMENTS

It is a pleasure to acknowledge L. Pietronero, M. Scattoni, S. Pellegrini, and Juan Ruiz-Lorenzo for useful comments and discussions, and J. M. Sancho for a critical reading of the first draft and interesting correspondence. This work has been partially supported by the European Union through a grant to M.A.M. (Grant No. ERBFMBICT960925), and by the TMR ''Fractals'' network under Project No. EMRXCT980183.

[1] See W. Horsthemke and R. Lefever, *Noise Induced Transitions* (Springer Verlag, Berlin, 1984), and references therein.
[2] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); K. Weisenfeld and F. Moss, Nature (London) **373**, 33 (1995).
[3] C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992); P. Pechukas and P. Hänggi, *ibid.* **73**, 2772 (1994).
[4] J. García-Ojalvo, A. Hernández-Machado, and J. M. Sancho, Phys. Rev. Lett. **71**, 1542 (1993); J. M. Parrondo, C. Van den Broeck, J. Buceta, and F. J. de la Rubia, Physica A **224**, 153 (1996).
[5] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).
[6] A. Becker and L. Kramer, Phys. Rev. Lett. **73**, 955 (1994).
[7] A. S. Pikovsky and J. Kurths, Phys. Rev. E **49**, 898 (1994).
[8] C. Van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997).
[9] W. Genovese, M. A. Muñoz, and J. M. Sancho, Phys. Rev. E **57**, R2495 (1998).
[10] G. Grinstein, M. A. Muñoz, and Y. Tu, Phys. Rev. Lett. **76**, 4376 (1996); Y. Tu, G. Grinstein, and M. A. Muñoz, *ibid.* **78**, 274 (1997).
[11] J. García-Ojalvo, J. M. R. Parrondo, J. M. Sancho, and C. Van den Broeck, Phys. Rev. E **54**, 6918 (1996).
[12] R. Müller, K. Lippert, A. Kühnel, and U. Behn, Phys. Rev. E **56**, 2658 (1997).
[13] S. Kim, S. H. Park, and C. S. Ryu, Phys. Rev. Lett. **78**, 1616 (1997).
[14] M. F. Thorpe and D. Beeman, Phys. Rev. B **14**, 188 (1976).
[15] L. Onsager, Phys. Rev. **75**, 117 (1944).
[16] D. Amit, *Field theory, The Renormalization Group, and Critical Phenomena* (World Scientific, Singapore, 1984).
[17] P. C. Hohenberg and B. J. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[18] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).
[19] M. San Miguel and R. Toral, in *Instabilities and Nonequilibrium Structures VI*, edited by E. Tirapegui and W. Zeller (Kluwer Academic, Dordrecht, 1997).
[20] An extended discussion of this point will be presented elsewhere.